USE OF ORDINARY DIFFERENTIAL EQUATIONS APPLIED TO MECHANICAL AND ELECTRICAL PROBLEMS WITH FIRST ORDER LINEAR SYSTEMS

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Abstract: *In recent years there has been an increased interest in the study of mathematics applied to engineering to study aspects of its teaching. In the present research work, a compilation of analytical concepts of differential equations applied to the solution of problems of electrical circuits, a branch of electrical engineering and mechanical engineering problems, specifical problems of spring-mass mechanical systems, through the description of the analytical solutions of direct current electrical circuits and spring-mass mechanical systems, applying differential equations. This method stands out among the diverse mathematical treatments used to solve this type of engineering problem, both for its demonstrative elegance and capacity to offer a total answer. Likewise, practical engineering examples related to the studied subject were presented and the results were obtained and analyzed by using the mathematical software, Wolfram Mathematica.*

Keywords: Differential Equations; L-R-C circuits; Mechanical problems; Wolfram Mathematica; Spring-mass systems.

1. INTRODUCTION

Throughout history people in their desire to find options or solutions to facilitate their daily lives have used the tools that the environment where they live has provided them, this implies the feasibility of implementing projects; through which great discoveries have been made achieving the main objective: which is to provide solutions. One of these many projects or findings, is in the area of mathematics, specifically differential equations with which you can model different situations or phenomena that over the years not everyone stopped to study, but those who did, made a deep study of these.

In recent years, mathematics is being applied in a large number of work sectors and its relevance is growing exponentially. Within mathematics, differential equations are a great tool for their countless applications in other disciplines [1,2].

Differential equations began to be studied for the reasons of change, with problems such as How long does it take? Why does it move? How fast does it change? All these questions led many people to carry out various studies. Concluding that the answer to them was differential equations [3–5].

What is a differential equation, what is a differential equation for, how is one applied in the engineering area, and will they be useful at work? Questions like these may arise when you are beginning to study Differential Equations; to answer the first question not only of order but also of importance it can be said that it is defined in general terms, a Differential Equation is an equation that involves the derivatives of a function with the function itself and/or the variables on which it depends [6,7]. In its applications, functions generally represent quantities and the derivatives are the rates of variation of these quantities.

A differential equation is classified according to its type, order, and linearity; by its type, it can be an ordinary differential equation or a partial derivative equation. The order of a differential equation is governed by the order of the highest order derivative in the equation; according to its linearity, it can be a linear or nonlinear equation. The above is a simple summary of the principles or basic concepts of differential equations, from them, more concepts are derived, and obviously, the most important part of them is how to solve them and their applications, the reason for this study [1,8,9]. The applications that have been mostly studied are the analysis of relationships between quantities and their rates of change, which are frequent in areas such as Physics, Biology, Engineering, or Economics [10–14]. There is a large number of complex real phenomena that can be formulated in terms of differential equations, which allows for predicting their future behaviors, among other things. Therefore, differential equations allowed these sciences to move from being empirical to being descriptive and predictive.

Differential equations are widely used in all branches of engineering to model physical phenomena. They are characterized by having one or more dependent variables related to one or more independent variables [15,16]. They are usually accompanied by initial conditions, such as boundary constraints or derivatives as boundary conditions, which delimit the problem at hand [3].

It must be known if the problem has a solution and also if the solution is unique, to evaluate the answer then traditional procedures are used. There are cases for which it is not possible to find a solution when there are complex geometries, therefore, it is necessary to resort to numerical methods to find the approximate answer. Numerical methods are widely used in the solution of mathematical models, their great diffusion is because they lead to simple algebraic expressions very easy to manipulate with modern computational tools, as is the case of Wolfram Mathematica [7,17], for the development of practical engineering problems.

In engineering, there are a great variety of situations that are modeled with differential equations, for example, applications in solid and fluid mechanics, hydrodynamic problems, mechanical vibrations (spring-mass mechanical systems) [13,18–20], diffusion processes, acoustic behavior of materials, heat transfer, electrical circuits, electromagnetism, and others [12,21,22]. Differential equations are a great mathematical tool to describe real situations or phenomena. Therefore, this research develops the description of the analytical solutions of electric circuits and mechanical spring-mass systems, applying differential equations to the solution of practical problems related to the subject of study with the help of Wolfram mathematical software.

2. DEVELOPMENT

To begin with, we will deal with applications in electrical circuits in the electrical area, based on principles and laws of electricity known as Kirchhoff's Law [10,22]. And likewise in mechanics, in problems of spring-mass systems, since this has to do with physics, which in turn deals with the investigation of the laws governing the behavior of the physical universe. And for the study of physics, Newton's laws are indispensable.

2.1. Analysis of L-R-C circuits

One of the applications of the second-order non-homogeneous linear ordinary differential equations arises in the study of electrical circuits after the application of *Kirchhoff's law and Kirchhoff's voltage law*. Analyzing the circuit in Figure 1, suppose that $I(t)$ is the current in the L-R-C series electric circuit where L, R, and C represent the inductance, resistance, and capacitance of the circuit, respectively.

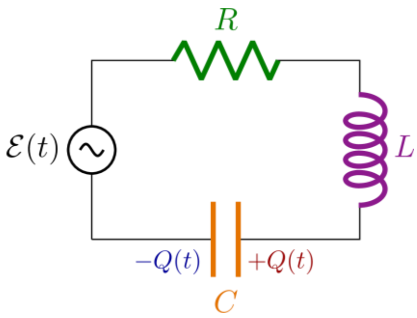

Figure 1. L-R-C circuits.

***Kirchhoff's Law**: The current entering a point in the circuit is equal to the current leaving the point.*
***Kirchhoff's voltage law**: The sum of the voltage changes around each loop in the circuit is zero.*

Table 1. L-R-C circuit voltage drop analysis

Circuit element	Voltage drops
Inductor	$L\frac{dI}{dt}$
Resistor	RI
Capacitor	$Q\frac{I}{C}$
Voltage source	$-V(t)$

Table 1 shows the analysis of the voltage drops across the circuit elements, which have been obtained from experimental data where Q is the capacitor charge and $dQ/dt = I$.
The main objective of this study is to model this physical situation of the L-R-C circuit with an initial value problem so that the current and load in the circuit can be determined. For convenience, the terminology used, and the dimensional analysis used is summarized in Table 2.

Table 2. Terminology and dimensional analysis

Electrical quantities	Units
Inductance (L)	Henrys (H)
Resistance (R)	Ohms (Ω)
Capacitance (C)	Farads (F)
Charge (Q)	Coulombs (C)
Current (I)	Amperes (A)

Applying Kirchhoff's law: the sum of the voltage drops across the circuit elements is equivalent to the voltage across the L-R-C series circuit in Figure 1, the physical principle necessary to derive the differential equation that models this circuit can be obtained and therefore, the following differential equation is obtained.

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t) \tag{1}$$

From the analysis of Table 1, the capacitor charge is shown and using this concept $dQ/dt = I$, can also be obtained $d^2Q/dt^2 = dI/dt$. Replacing these definitions in Equation 1, this equation becomes:

$$L\frac{dQ^2}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}Q = E(t) \tag{2}$$

This differential equation can be solved by the method of undetermined coefficients or the method of parameter variation. Therefore, if the initial load and current are $Q(0) = Q_0$ and $I(0) = Q'(0) = I_0$, then we must solve the initial-value problem for the charge $Q(t)$.

$$\begin{cases} L\dfrac{dQ^2}{dt^2} + R\dfrac{dI}{dt} + \dfrac{1}{C}Q = E(t) \\ Q(0) = Q_0, \quad I(0) = \dfrac{dQ}{dt}(0) = I_0 \end{cases} \tag{3}$$

This solution can then be differentiated to find the current $I(t)$.

2.2. Analysis of spring-mass mechanical systems

Consider a spring of length l_0 suspended vertically from a rigid support. If we hang a mass m from it, the spring will elongate by an amount, called elongation, which we denote by Δl (see Figure 2).

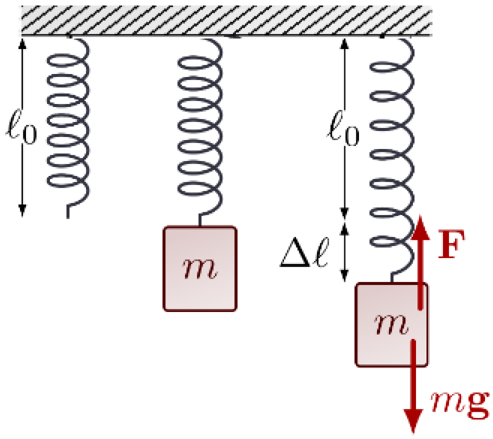

Figure 2. Representation of a spring-mass mechanical system

The displacement of a mass attached to the end of a spring can be modeled with a second-order linear differential equation with constant coefficients. Similarly, this situation can also be expressed as a system of first-order ordinary differential equations.

A spring has a natural length l_0. When a mass is attached to the spring, it is stretched s units past its natural length to the equilibrium position $x = 0$. When the system is put into motion, the displacement from $x = 0$ at time t is given by $x(t)$.

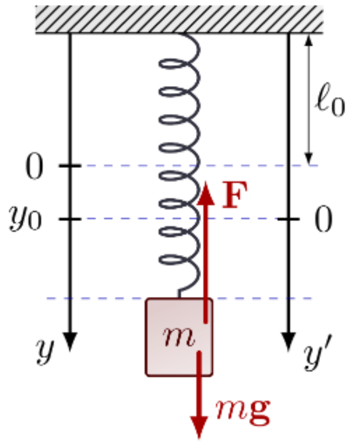

Figure 3. Forces acting on the spring-mass system

By Newton's second law of motion, $F = ma = md^2x/dt^2$, where m represents mass and $a = g$ represents acceleration (see Figure 3). If we assume that there are no other forces acting on the mass, then we determine the differential equation that models this situation in the following way:

$$m\frac{d^2x}{dt^2} = \sum(\,forces\ \ acting\ on\ the\ system)$$

$$m\frac{d^2x}{dt^2} = -k(s+x)+mg \tag{4}$$

$$m\frac{d^2x}{dt^2} = -ks-kx+mg$$

At equilibrium $ks = mg$, so after simplification, we obtain the differential equation:

$$m\frac{d^2x}{dt^2} = -ks \quad or \quad m\frac{d^2x}{dt^2}+ks = 0 \tag{5}$$

The two initial conditions that are used with this problem are the initial displacement $x(0) = \alpha$ and initial velocity $dx/dt\,(0) = \beta$. Hence, the function $x(t)$ that describes the displacement of the mass with respect to the equilibrium position is found by solving the initial-value problem:

$$\begin{cases} m\dfrac{d^2x}{dt^2}+ks=0, \\ x(0)=\alpha, \quad \dfrac{dx}{dt}(0)=\beta \end{cases} \tag{6}$$

The differential equation in initial-value problem (6) disregards all retarding forces acting on the motion of the mass.

Analyzing that if there is no external forcing function, the system is called *Simple Harmonic Motion* and the second order differential equation that models this situation is:

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=0 \tag{7}$$

where m is the mass attached to the end of the spring, c is the damping coefficient and k is the spring constant found with *Hooke's law*. According to Hooke's law, the spring exerts a restoring force in the upward direction that is proportional to the displacement of the spring [19,20].

Hooke's law: $F = ks$, where $k > 0$ is the constant of proportionality or spring constant and s is the displacement of the spring.

The Equation 7 is transformed into a system of equations by letting $x' = y$ so that $y' = x'' = -\frac{k}{m}x - \frac{c}{m}x'$ and then solving the differential equation for x''. After substitution, we have the system:

$$\begin{cases} \dfrac{dx}{dt}=y, \\ \dfrac{dy}{dt}=-\dfrac{k}{m}x-\dfrac{c}{m}y \end{cases} \tag{8}$$

3. APPLICATIONS

For the illustration of the applications of the differential equations of the physical engineering models studied in this research, they were taken from the text Introductory Differential Equations by Abell & Braselton [3,4].

3.1. Two-loop L-R-C circuit exercise

Find Q(t), I(t), $I_1(t)$ and $I_2(t)$ in the L-R-C circuit with two loops (see Figure 4) given that $R_1 = R_2 = 1\Omega$, $C = 1$F, $L = 1$H and $V(t) = e^{-t}$ V if $Q(0) = 3$C and $I_2(0) = 1$A.

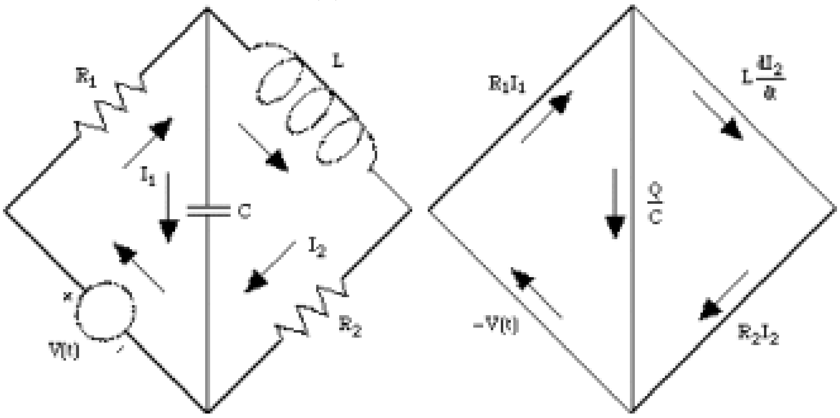

Figure 4. L-R-C circuit with two loops

Solution:

In this case, the current through the capacitor is equivalent to $I_1 - I_2$. Summing the voltage drops around each loop, we have:

$$\begin{cases} R_1 I_1 + \frac{1}{C} Q - V(t) = 0, \\ L \frac{dI_2}{dt} + R_2 I_2 - \frac{1}{C} Q = 0 \end{cases} \tag{9}$$

Solving the first equation for I_1 we find that $I_1 = \frac{1}{R_1} V(t) - \frac{1}{R_1 C} Q$ and using the relationship $dQ/dt = I = I_1 - I_2$ we have the following system:

$$\begin{cases} \frac{dQ}{dt} = -\frac{1}{R_1 C} Q - I_2 + \frac{1}{R_2} V(t), \\ \frac{dI_2}{dt} = \frac{1}{LC} Q - \frac{R_2}{L} I_2 \end{cases} \tag{10}$$

Replacing the problem data in Equation 10, the inhomogeneous system that models this circuit is:

$$\begin{cases} \frac{dQ}{dt} = -Q - I_2 + e^{-t}, \\ \frac{dI_2}{dt} = Q - I_2 \end{cases}$$

Given the initial conditions $Q(0) = 1$ and $I_2(0) = 3$, which the problem poses as data and using the method of variation of parameters to solve the problem, therefore, in matrix form this system is:

$$\begin{pmatrix} dQ/dt \\ dI_2/dt \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} Q \\ I_2 \end{pmatrix} + \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}$$

The eigenvalues of the corresponding homogeneous system are $\lambda_{1,2} = -1 \pm i$, and an eigenvector corresponding to λ_1 is $V_1 = \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} i$

Two linearly independent solutions of the corresponding homogeneous system are:

$$\chi_1(t) = \begin{pmatrix} Q(t) \\ I_2(t) \end{pmatrix} = e^{-t} \cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix} - e^{-t} \sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\chi_1(t) = \begin{pmatrix} -e^{-t} \sin t \\ e^{-t} \cos t \end{pmatrix}$$

and

$$\chi_2(t) = \begin{pmatrix} Q(t) \\ I_2(t) \end{pmatrix} = e^{-t} \cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - e^{-t} \sin t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\chi_2(t) = \begin{pmatrix} e^{-t} \cos t \\ e^{-t} \sin t \end{pmatrix}$$

so a fundamental matrix is:

$$\Phi(t) = \begin{pmatrix} -e^{-t} \sin t & e^{-t} \cos t \\ e^{-t} \cos t & e^{-t} \sin t \end{pmatrix} \quad \text{and its transposition:}$$

$$\Phi^{-1}(t) = \begin{pmatrix} -e^{t} \sin t & e^{t} \cos t \\ e^{t} \cos t & e^{t} \sin t \end{pmatrix}$$

Therefore,

$$\chi(t)=\Phi(t)\Phi^{-1}(0)\chi(0)+\Phi(t)\int_0^t\Phi^{-1}(u)F(u)du$$

$$\chi(t)=\begin{pmatrix}-e^{-t}\sin t & e^{-t}\cos t\\ e^{-t}\cos t & e^{-t}\sin t\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ 3\end{pmatrix}+\begin{pmatrix}-e^{-t}\sin t & e^{-t}\cos t\\ e^{-t}\cos t & e^{-t}\sin t\end{pmatrix}\times\int_0^t\begin{pmatrix}-e^{u}\sin u & e^{u}\cos u\\ e^{u}\cos u & e^{u}\sin u\end{pmatrix}\begin{pmatrix}e^{-u}\\ 0\end{pmatrix}du$$

$$\chi(t)=\begin{pmatrix}-e^{-t}\sin t & e^{-t}\cos t\\ e^{-t}\cos t & e^{-t}\sin t\end{pmatrix}\begin{pmatrix}3\\ 1\end{pmatrix}+\begin{pmatrix}-e^{-t}\sin t & e^{-t}\cos t\\ e^{-t}\cos t & e^{-t}\sin t\end{pmatrix}\times\int_0^t\begin{pmatrix}-\sin u\\ \cos u\end{pmatrix}du$$

$$\chi(t)=\begin{pmatrix}e^{-t}\cos t-3e^{-t}\sin t\\ 3e^{-t}\cos t+e^{-t}\sin t\end{pmatrix}+\begin{pmatrix}-e^{-t}\sin t & e^{-t}\cos t\\ e^{-t}\cos t & e^{-t}\sin t\end{pmatrix}\begin{pmatrix}\cos t-1\\ \sin t\end{pmatrix}$$

$$\chi(t)=\begin{pmatrix}e^{-t}\cos t-3e^{-t}\sin t\\ 3e^{-t}\cos t+e^{-t}\sin t\end{pmatrix}+\begin{pmatrix}e^{-t}\sin t\\ e^{-t}-e^{-t}\cos t\end{pmatrix}$$

$$\chi(t)=\begin{pmatrix}e^{-t}\cos t-2e^{-t}\sin t\\ 2e^{-t}\cos t+e^{-t}+e^{-t}\sin t\end{pmatrix}$$

This would be the result, because:

$$\frac{dQ}{dt}=I \quad and$$

$$Q(t)=e^{-t}\cos t-2e^{-t}\sin t$$

which, differentiation yields:

$$I(t)=-3e^{-t}\cos t+e^{-t}\sin t$$

Also, because:

$$I_1(t)=I(t)+I_2(t),$$

$$I_1(t)=-e^{-t}\cos t+2e^{-t}\sin t+e^{-t}$$

Wolfram Mathematica solution with DSolve:

The Wolfram programming code for the solution of the problems posed in this section is attached as supplementary material.

```
In[•]:= Solution = DSolve[{D[q[t], t] == -q[t] - i2[t] + Exp[-t], D[i2[t], t] == q[t] -
            i2[t], q[0] == 3, i2[0] == 1}, {q[t], i2[t]}, t] // Simplify
Out[•]= {{i2[t] → e^-t (1 + 3 Sin[t]), q[t] → 3 e^-t Cos[t]}}

In[•]:= q[t_] = 3 e^-t Cos[t];
        i2[t_] = e^-t (1 + 3 Sin[t]);

In[•]:= i[t_] = D[q[t], t]
Out[•]= -3 e^-t Cos[t] - 3 e^-t Sin[t]

In[•]:= i1[t_] = i[t] + i2[t]
Out[•]= -3 e^-t Cos[t] - 3 e^-t Sin[t] + e^-t (1 + 3 Sin[t])
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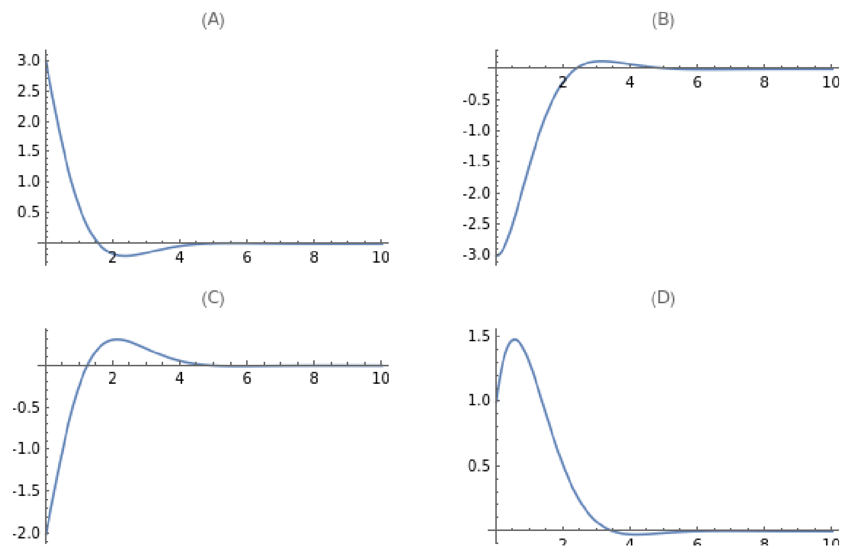
Similarly, the graphical solution is presented using Wolfram.



3.2. Exercise on a mechanical spring-mass system with *Simple Harmonic Motion*

Solve the system of differential equations to find the displacement of the mass if $m = 1$, $c = 0$, and $k = 1$.

Solution: In this case, the system is:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x$$

The eigenvalues are solutions of:

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \quad so:$$

$$\lambda_{1,2} = \pm i$$

An eigenvector corresponding to λ_1 is:

$$V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} i$$

so two linearly independent solutions are:

$$\chi_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t$$

$$\chi_1(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \qquad and$$

$$\chi_2(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t$$

$$\chi_2(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

A general solution is:

$$\chi(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \chi_1(t) + c_2 \chi_2(t)$$

$$\chi(t) = \begin{pmatrix} c_1 \cos t + c_2 \sin t \\ c_2 \cos t - c_1 \sin t \end{pmatrix}$$

Wolfram Mathematica solution with DSolve:

```
In[•]:= res = DSolve[{D[x[t], t] == y[t], D[y[t], t] == -x[t], x[0] == x0, y[0] ==
        y0}, {x[t], y[t]}, t]
Out[•]= {{x[t] → x0 Cos[t] + y0 Sin[t], y[t] → y0 Cos[t] - x0 Sin[t]}}
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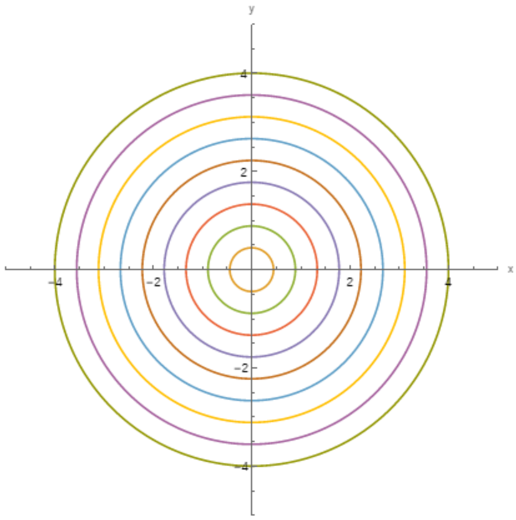

4. CONCLUSIONS

The use of differential equations in the modeling of mechanical systems and electrical circuits facilitates their resolution, both manually and with the help of mathematical software, such as Wolfram, which was used to compare the solutions.

Wolfram mathematical software is a powerful tool for solving differential equations, which helped to validate the results of the problems posed. It also makes it easier to obtain a graphical illustration of the solution to the problems, which can often be very complex to graph a solution to a differential equation.

Finally, it is concluded that differential equations have had an influence on many areas of life. In addition, they have served, serve and will serve as long as there are problems that can be solved with them. Differential equations are then the fruit of a long process of study, so learning them, understanding them and applying them is to put the result of that study to good use.

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